ON OSCILLATION OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract. An interrelationship is found between the accumulation points of zeros of non-trivial solutions of $f''+Af=0$ and the boundary behavior of the analytic coefficient A in the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$.

It is also shown that the geometric distribution of zeros of any non-trivial solution of $f''+Af=0$ is severely restricted if

$$|A(z)|(1-|z|^2)^2 \le 1 + C(1-|z|), \quad z\in\mathbb{D}, \tag{$\star$}$$

for any constant $0<C<\infty$. These considerations are related to the open problem whether ($\star$) implies finite oscillation for all non-trivial solutions.

1. Introduction

The following result plays a decisive role in the oscillation theory of solutions of linear differential equation

$$f''+Af=0 \tag{1.1}$$

in the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. If A is an analytic function in $\mathbb{D}$ for which

$$|A(z)|(1-|z|^2)^2 \le 1, \quad z\in\mathbb{D}, \tag{1.2}$$

then each non-trivial solution f of (1.1) vanishes at most once in $\mathbb{D}$. This statement corresponds to the well-known result of Z. Nehari [11, Theorem 1], which provides a sufficient condition for injectivity of any locally univalent meromorphic function w in $\mathbb{D}$ in terms of the size of its Schwarzian derivative

$$S_w = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2 = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2.$$

The corresponding necessary condition was invented by W. Kraus [9], and rediscovered by Nehari [11, Theorem 1] some years later. In the setting of differential equations it states that, if A is analytic in $\mathbb{D}$, and each solution f of (1.1) vanishes at most once in $\mathbb{D}$, then $|A(z)|(1-|z|^2)^2 \le 3$ for all $z\in\mathbb{D}$. An important discovery of B. Schwarz [15, Theorems 3–4] shows that the condition

$$\sup_{z\in\mathbb{D}} |A(z)|(1-|z|^2)^2 < \infty,$$

which allows non-trivial solutions of (1.1) to have infinitely many zeros in $\mathbb{D}$, is both necessary and sufficient for zeros of all non-trivial solutions to be separated with respect to the hyperbolic metric.

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Our first objective is to consider the interrelationship between the accumulation points of zeros of non-trivial solutions f of (1.1) and the boundary behavior of the coefficient A. The second objective is a question of more specific nature. We consider differential equations (1.1) in which the growth of the coefficient barely exceeds the bound (1.2) that ensures finite oscillation.

2. Results

2.1. **Accumulation points of zeros of solutions.** The point of departure is a result, which associates the zero-sequences of non-trivial solutions of (1.1) to the boundary behavior of the coefficient. This theorem sets the stage for more profound oscillation theory.

Theorem 1. *Let A be an analytic function in $\mathbb{D}$, and let $\zeta \in \partial\mathbb{D}$.*

If there exists a sequence $\{w_n\} \subset \mathbb{D}$ converging to ζ, such that

$$|A(w_n)|(1-|w_n|^2)^2 \to c \tag{2.1}$$

for some $c \in (3, \infty]$, then for each $\delta > 0$ there exists a non-trivial solution of (1.1) *having two distinct zeros in $D(\zeta, \delta) \cap \mathbb{D}$.*

Conversely, if for each $\delta > 0$ there exists a non-trivial solution of (1.1) *having two distinct zeros in $D(\zeta, \delta) \cap \mathbb{D}$, then there exists a sequence $\{w_n\} \subset \mathbb{D}$ converging to ζ such that* (2.1) *holds for some $c \in [1, \infty]$.*

We point out that (2.1) with $c \in (3, \infty]$ does not necessarily imply infinite oscillation for any non-trivial solution of (1.1), see Example 2 below.

The proof of Theorem 1 is based on theorems by Nehari and Kraus, and on a principle of localization. One of the key factors is an application of a suitable family of conformal maps under which the image of $\mathbb{D}$ has a smooth boundary, that intersects $\partial\mathbb{D}$ precisely on an arc centered at $\zeta \in \partial\mathbb{D}$. The second assertion of Theorem 1 is implicit in the proof of [11, Theorem 1], and follows directly from the following property: if $z_1, z_2 \in \mathbb{D}$ are two distinct zeros of a non-trivial solution f of (1.1), then there exists a point $w \in \mathbb{D}$, which belongs to the hyperbolic geodesic going through z_1 and z_2, such that $|A(w)|(1-|w|^2)^2 > 1$.

2.2. **Chuaqui-Stowe question.** Schwarz [15, Theorem 1] supplemented the oscillation theory by proving that, if there exists a constant $0 < R < 1$ such that

$$|A(z)|(1-|z|^2)^2 \le 1, \quad R < |z| < 1,$$

then each non-trivial solution of (1.1) has at most finitely many zeros. Schwarz also gave an example [15, p. 162] showing that the constant one in the right-hand side of (1.2) is best possible. That is, for each $\gamma > 0$, the functions

$$A(z) = \frac{1+4\gamma^2}{(1-z^2)^2} \quad \text{and} \quad f(z) = \sqrt{1-z^2}\,\sin\left(\gamma \log\frac{1+z}{1-z}\right)$$

satisfy (1.1), while f has infinitely many (real) zeros in $\mathbb{D}$. Example 1 below shows the sharpness of Kraus' result.

M. Chuaqui and D. Stowe [4, Theorem 5] constructed an example showing that for each continuous function $\varepsilon\colon [0,1) \to [0,\infty)$ satisfying $\varepsilon(r) \to \infty$ as $r \to 1^-$, there exists an analytic function A such that

$$|A(z)|(1-|z|^2)^2 \le 1 + \varepsilon(|z|)(1-|z|), \quad z \in \mathbb{D}, \tag{2.2}$$

while (1.1) admits a non-trivial solution having infinitely many zeros. In other words, if $\varepsilon(r)(1-r)$ in (2.2) does not decay to zero as fast as linear rate as $r \to 1^-$,

then non-trivial solutions of (1.1) may have infinitely many zeros. This is in contrast to the case of real differential equations (1.1) on the open interval $(-1,1)$, since then $\varepsilon(r)=(1-r)^{-1}(-\log(1-r))^{-2}$ distinguishes finite and infinite oscillation, see [2, 4] for more details. Chuaqui and Stowe [4, p. 564] left open a question whether

$$|A(z)|(1-|z|^2)^2 \leq 1+C(1-|z|), \quad z\in\mathbb{D}, \tag{2.3}$$

with some or any $0<C<\infty$, implies finite oscillation for all non-trivial solutions of (1.1). The following results do not give a complete answer to this question, however, they indicate that both the growth and the zero distribution of non-trivial solutions of (1.1) are severely restricted if (2.3) holds for some $0<C<\infty$.

2.2.1. *Growth of solutions.* An analytic function f in $\mathbb{D}$ belongs to the growth space H^∞_α for $0\leq\alpha<\infty$, if

$$\|f\|_{H^\infty_\alpha}=\sup_{z\in\mathbb{D}}|f(z)|(1-|z|^2)^\alpha<\infty.$$

It is known that the growth of A restricts the growth of solutions of (1.1). If $A\in H^\infty_2$, then there exists a constant $p=p(\|A\|_{H^\infty_2})$ with $0\leq p<\infty$ such that all solutions f of (1.1) satisfy $f\in H^\infty_p$. This result can be deduced by using classical comparison theorems [14, Example 1], Gronwall's lemma [8, Theorem 4.2] or successive approximations [7, Theorem I], for example. We conclude this result by means of straightforward integration. See Example 3 for sharpness discussion.

Theorem 2. *Let A be an analytic in $\mathbb{D}$ such that $|A(z)|(1-|z|^2)^2\leq K+\varepsilon(|z|)$ for all $z\in\mathbb{D}$, where $0\leq K<\infty$ is a constant, and $\varepsilon(|z|)\to 0$ as $|z|\to 1^-$. Then all solutions of* (1.1) *belong to H^∞_p for any $(\sqrt{1+K}-1)/2<p<\infty$.*

2.2.2. *Separation of zeros of solutions.* The following result establishes a connection between the separation of zeros of non-trivial solutions of (1.1) and the growth of the coefficient function A; compare to [15, Theorems 3 and 4].

If z_1,z_2 are two distinct points in $\mathbb{D}$, then the pseudo-hyperbolic distance $\varrho_p(z_1,z_2)$ and the hyperbolic distance $\varrho_h(z_1,z_2)$ between z_1 and z_2 are given by

$$\varrho_p(z_1,z_2)=\big|\varphi_{z_1}(z_2)\big|, \quad \varrho_h(z_1,z_2)=\frac{1}{2}\log\frac{1+\varrho_p(z_1,z_2)}{1-\varrho_p(z_1,z_2)},$$

where $\varphi_a(z)=(a-z)/(1-\overline{a}z)$, $a\in\mathbb{D}$. Moreover, let $\xi_h(z_1,z_2)$ denote the hyperbolic midpoint between z_1 and z_2. Correspondingly,

$$\Delta_p(a,r)=\big\{z\in\mathbb{D}:\varrho_p(z,a)<r\big\}, \quad \Delta_h(a,r)=\big\{z\in\mathbb{D}:\varrho_h(z,a)<r\big\},$$

are the pseudo-hyperbolic and hyperbolic discs of radius $r>0$ centered at $a\in\mathbb{D}$, respectively.

Theorem 3. *Let A be an analytic function in $\mathbb{D}$.*

If the coefficient A satisfies (2.3) *for some $0<C<\infty$, then the hyperbolic distance between any distinct zeros $z_1,z_2\in\mathbb{D}$ of any non-trivial solution of* (1.1), *for which $1-|\xi_h(z_1,z_2)|<1/C$, satisfies*

$$\varrho_h(z_1,z_2)\geq\log\frac{2-C^{1/2}(1-|\xi_h(z_1,z_2)|)^{1/2}}{C^{1/2}(1-|\xi_h(z_1,z_2)|)^{1/2}}. \tag{2.4}$$

Conversely, if there exists a constant $0<C<\infty$ such that any two distinct zeros $z_1,z_2\in\mathbb{D}$ of any non-trivial solution of (1.1), *for which $1-|\xi_h(z_1,z_2)|<1/C$, satisfies* (2.4), *then*

$$|A(z)|(1-|z|^2)^2\leq 3\left(1+\Psi_C(|z|)(1-|z|)^{1/3}\right), \quad 1-|z|<(8C)^{-1}, \tag{2.5}$$

where Ψ_C *is positive, and satisfies* $\Psi_C(|z|) \longrightarrow \big(2\,(8C)^{1/3}\big)^+$ *as* $|z| \to 1^-$.

Concerning Theorem 3 note that, if $1 - |\xi_h(z_1, z_2)| < 1/C$, then (2.4) implies

$$\varrho_h(z_1, z_2) \geq \frac{1}{2}\log\frac{1}{C} + \frac{1}{2}\log\frac{1}{1 - |\xi_h(z_1, z_2)|},$$

and hence $\varrho_h(z_1, z_2)$ is large whenever $\xi_h(z_1, z_2)$ is close to the boundary $\partial\mathbb{D}$.

If the coefficient A satisfies (2.3) for some $0 < C < \infty$, and f_1 and f_2 are linearly independent solutions of (1.1), then the quotient $w = f_1/f_2$ is a normal function (in the sense of Lehto and Virtanen) by [16, Corollary, p. 328]. As a direct consequence we deduce the following corollary, which states that the zero-sequences of f_1 and f_2 are hyperbolically separated from each other, see also Example 4 below.

Corollary 4. *Let* A *be an analytic function in* $\mathbb{D}$*, which satisfies* (2.3) *for some* $0 < C < \infty$*, and let* $\{z_n\}$ *and* $\{\zeta_m\}$ *be the zero-sequences of two linearly independent solutions* f_1 *and* f_2 *of* (1.1). *Then, there is a constant* $\delta = \delta(f_1, f_2)$ *such that* $\varrho_h(z_n, \zeta_m) > \delta > 0$ *for all* n *and* m.

The following result shows that, if (2.3) does not imply finite oscillation for non-trivial solutions of (1.1), then infinite zero-sequences tend to $\partial\mathbb{D}$ tangentially. Any disc $D(\zeta, 1 - |\zeta|)$ for $\zeta \in \mathbb{D}$, which is internally tangent to $\mathbb{D}$, is called a horodisc. Note that Theorem 5 remains valid in the limit case $C = 0$ by the classical theorems of Nehari and Kraus.

Theorem 5. *Let* A *be an analytic function in* $\mathbb{D}$.

If A *satisfies* (2.3) *for some* $0 < C < \infty$*, then any non-trivial solution of* (1.1) *has at most one zero in any Euclidean disc* $D(\zeta, (1 + C)^{-1})$ *for* $|\zeta| \leq C/(1 + C)$.

Conversely, if there exists $0 < C < \infty$ *such that any non-trivial solution of* (1.1) *has at most one zero in any Euclidean disc* $D(\zeta, (1 + C)^{-1})$ *for* $|\zeta| \leq C/(1 + C)$*, then*

$$|A(z)|(1 - |z|^2)^2 \leq 3\Big(1 + \Psi_C(|z|)\,(1 - |z|)\Big), \quad \frac{C}{1 + C} < |z| < 1,$$

where Ψ_C *is positive, and satisfies* $\Psi_C(|z|) \longrightarrow (2C)^+$ *as* $|z| \to 1^-$.

2.2.3. *Geometric distribution of zeros of solutions.* The set

$$Q = Q(I) = \big\{re^{i\theta} : e^{i\theta} \in I,\, 1 - |I| \leq r < 1\big\}$$

is called a Carleson square based on the arc $I \subset \partial\mathbb{D}$, where $|I| = \ell(Q)$ denotes the normalized arc length of I (i.e., $|I|$ is the Euclidean arc length of I divided by 2π).

Theorem 6. *If* A *is an analytic function in* $\mathbb{D}$ *such that* (2.3) *holds for some* $0 < C < \infty$*, then the zero-sequence* $\{z_n\}$ *of any non-trivial solution of* (1.1) *satisfies*

$$\sum_{z_n \in Q} (1 - |z_n|)^{1/2} \leq K\,\ell(Q)^{1/2}, \tag{2.6}$$

for any Carleson square Q. *Here* $K = K(C)$ *with* $0 < K < \infty$ *is a constant independent of* f.

If A is analytic in $\mathbb{D}$ and satisfies (2.3) for some $0 < C < \infty$, then the zero-sequence of any non-trivial solution of (1.1) is interpolating by Theorem 6, because the zero-sequences are separated by the classical result of Schwarz.

3. Examples

We turn to consider some non-trivial examples, the first of which shows the sharpness of Kraus' result.

Example 1. Let
$$A(z) = -\frac{3}{4(1-z)^2}, \quad z \in \mathbb{D}.$$
A solution base $\{f_1, f_2\}$ of (1.1) is given by the non-vanishing functions
$$f_1(z) = (1-z)^{-1/2}, \quad f_2(z) = (1-z)^{3/2}, \quad z \in \mathbb{D}.$$
Let f be any non-trivial solution of (1.1). If f is linearly dependent to f_1 or f_2, then f is non-vanishing. Otherwise, there exist $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ such that $f(z) = \alpha f_1(z) + \beta f_2(z)$, and $f(z) = 0$ if and only if $(1-z)^2 = -\alpha/\beta$. This equation has two solutions $z_1, z_2 \in \mathbb{C}$, and only one zero of f, say z_1, satisfies $\operatorname{Re} z_1 < 1$. This follows from the fact that $1 - z_1 = z_2 - 1$. Consequently, each solution of (1.1) has at most one zero in $\mathbb{D}$, while the coefficient function A satisfies $|A(z)|(1-|z|^2)^2 \to 3$ as $z \to 1^-$ along the positive real axis.

To conclude that (2.1) with $c \in (3, \infty]$ does not imply infinite oscillation for any solution of (1.1), we recall Hille's example [15, Eq. (2.12)]. The same example is also used in [3, Example 20].

Example 2. Let $A(z) = a/(1-z^2)^2$, where $-\infty < a < 0$ is a real parameter. If
$$f_1(z) = \sqrt{1-z^2}\left(\frac{1-z}{1+z}\right)^{\frac{1}{2}\sqrt{1-a}}, \quad f_2(z) = \sqrt{1-z^2}\left(\frac{1-z}{1+z}\right)^{-\frac{1}{2}\sqrt{1-a}},$$
then $\{f_1, f_2\}$ is a solution base of (1.1) of non-vanishing functions. Let f be any non-trivial solution of (1.1). If f is linearly dependent to f_1 or f_2, then f is non-vanishing. Otherwise, there exist $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ such that $f = \alpha f_1 + \beta f_2$. In this case $f(z) = 0$ if and only if
$$\frac{-\beta}{\alpha} = \left(\frac{1-z}{1+z}\right)^{\sqrt{1-a}}. \tag{3.1}$$
Since $z \mapsto (1-z)/(1+z)$ maps $\mathbb{D}$ onto the right half-plane, $\sqrt{1-a} \leq 4$ ensures that each solution of (1.1) has at most two zeros in $\mathbb{D}$, see also [15, p. 174]. Further, if $2 < \sqrt{1-a}$, then there exists a solution having exactly two zeros in $\mathbb{D}$. In particular, if $-\beta/\alpha$ is real and strictly negative, then $\alpha f_2 + \beta f_2$ has two zeros in $\mathbb{D}$ by (3.1), and these zeros are complex conjugate numbers in $\mathbb{D}$. Note that $2 < \sqrt{1-a} \leq 4$ if and only if $-15 \leq a < -3$, and then $|A(x)|(1-|x|^2)^2 = |a| > 3$ for all $x \in (0,1)$.

We fix $a = -8$, and discuss the zeros of the solution $f = f_1 + kf_2$ for $k > 0$. By (3.1), the zeros of f in $\mathbb{D}$ are solutions of $-k = (1-z)^3/(1+z)^3$. We conclude that f has exactly two zeros in $\mathbb{D}$ given by
$$z_1 = \frac{1 - \sqrt[3]{k}\,\exp(i\pi/3)}{1 + \sqrt[3]{k}\,\exp(i\pi/3)}, \quad z_2 = \frac{1 - \sqrt[3]{k}\,\exp(-i\pi/3)}{1 + \sqrt[3]{k}\,\exp(-i\pi/3)}.$$
If $k \to 0^+$, then z_1 and $z_2 = \overline{z}_1$ converge to $z = 1$ inside the unit disc. Now, for each $\delta > 0$ there exists a solution of (1.1) having two distinct zeros in $D(1,\delta) \cap \mathbb{D}$.

The following example concerns the sharpness of Theorem 2.

Example 3. Let $0 \le K < \infty$, and let A be the analytic function

$$A(z) = -\frac{K + 4\sqrt{1+K}\left(\log\frac{e}{1-z}\right)^{-1}}{4(1-z)^2}, \quad z \in \mathbb{D}.$$

Now

$$|A(z)|(1-|z|^2)^2 \le K + \frac{4\sqrt{1+K}}{\log\frac{e}{1-|z|}}, \quad z \in \mathbb{D}.$$

However, the analytic function

$$f(z) = \frac{1}{(1-z)^{(\sqrt{1+K}-1)/2}} \log\frac{e}{1-z}$$

is a solution of (1.1) such that $f \notin H_p^\infty$ for $p = (\sqrt{1+K}-1)/2$.

The following example shows that the number of zeros of a solution of (1.1) may be larger than any pregiven number, while the coefficient function A satisfies (2.3) for some sufficiently large $0 < C < \infty$. See [1] for similar examples concerning the cases of $A \in H_0^\infty$ and $A \in H_1^\infty$. Before the example, we recall some basic properties of the Legendre polynomials $P_0, P_1, P_2, \dots$, which can be recovered from Bonnet's recursion formula

$$nP_n(z) = (2n-1)zP_{n-1}(z) - (n-1)P_{n-2}(z), \quad P_0(z) = 1, \quad P_1(z) = z.$$

For every $n \in \mathbb{N}$, the Legendre polynomial P_n is known to have n distinct zeros in the interval $(-1,1)$, and P_n is a solution of Legendre's differential equation

$$(1-z^2)P_n''(z) - 2zP_n'(z) + n(n+1)P_n(z) = 0, \quad z \in \mathbb{D}, \quad n \in \mathbb{N}\cup\{0\}. \tag{3.2}$$

Example 4. Let P_n be the Legendre polynomial for $n \in \mathbb{N}\cup\{0\}$. By (3.2),

$$P_n''(z) + a_1(z)P_n'(z) + a_0(z)P_n(z) = 0, \quad a_1(z) = \frac{-2z}{1-z^2}, \quad a_0(z) = \frac{n(n+1)}{1-z^2},$$

for any $z \in \mathbb{D}$. Define $b(z) = -(1/2)\log(1-z^2)$ for $z \in \mathbb{D}$, and note that then b is a primitive of $-a_1/2$. According to [10, p. 74] the analytic function

$$f(z) = P_n(z)e^{-b(z)} = P_n(z)\,(1-z^2)^{1/2}, \quad z \in \mathbb{D},$$

which is bounded and has precisely n zeros in $\mathbb{D}$, is a solution of (1.1) with

$$A(z) = a_0(z) - \frac{1}{4}\left(a_1(z)\right)^2 - \frac{1}{2}a_1'(z) = \frac{1 + n(n+1)(1-z^2)}{(1-z^2)^2}, \quad z \in \mathbb{D}.$$

Let us consider the case $n = 0$ more closely. It is easy to verify that a solution base $\{f_1, f_2\}$ of (1.1) with $A(z) = (1-z^2)^{-2}$ is given by the non-vanishing functions

$$f_1(z) = (1-z^2)^{1/2}, \quad f_2(z) = (1-z^2)^{1/2}\log\frac{1+z}{1-z}, \quad z \in \mathbb{D}.$$

Let $0 < \alpha < \infty$. Then

$$f(z) = f_1(z) - \frac{1}{\alpha}f_2(z) = (1-z^2)^{1/2}\left(1 - \frac{1}{\alpha}\log\frac{1+z}{1-z}\right)$$

is also a solution of (1.1). Evidently, the point $z = z(\alpha)$ is a zero of f if and only if $z(\alpha) = (e^\alpha - 1)/(e^\alpha + 1)$. Since $z(\alpha)$ is a continuous function of $0 < \alpha < \infty$, we conclude that there exists a solution base $\{f_1 - f_2/\alpha_1, f_1 - f_2/\alpha_2\}$ of (1.1), where $0 < \alpha_1 < \alpha_2 < \infty$, such that the hyperbolic distance $\varrho_h(z(\alpha_1), z(\alpha_2))$ is smaller than any pregiven number. In particular, the constant $\delta > 0$ in Corollary 4 depends on the choice of linearly independent solutions, even if A satisfies (2.3) for $C = 0$.

By considering similar examples one can investigate the sharpness of the second assertions of Theorems 3 and 5. Details are left for the interested reader.

We next offer two concrete examples of equations whose solutions admit infinite oscillation, but the coefficient satisfies

$$|A(z)|(1-|z|^2)^2 \leq 1+\varepsilon(|z|), \quad z \in \mathbb{D}, \tag{3.3}$$

where $\varepsilon(|z|)$ decays to zero slower than the linear rate as $|z| \to 1^-$. The following example is similar to [3, Example 12].

Example 5. Let p be a locally univalent analytic function in $\mathbb{D}$. The functions

$$f_1(z) = \big(p'(z)\big)^{-1/2} \sin p(z), \quad f_2(z) = \big(p'(z)\big)^{-1/2} \cos p(z), \quad z \in \mathbb{D},$$

are linearly independent solutions of (1.1) with $A = (p')^2 + S_p/2$. We consider the equations (1.1) with $A = A_1$ and $A = A_2$ induced by

$$p_1(z) = \log\left(\log \frac{e^e}{1-z}\right), \quad p_2(z) = \left(\log \frac{e}{1-z}\right)^q,$$

where $0 < q < 1$. In the first case

$$A_1(z)(1-z)^2 = \frac{1}{4}\,\frac{5+\left(\log\frac{e^e}{1-z}\right)^2}{\left(\log\frac{e^e}{1-z}\right)^2},$$

and it follows that (3.3) holds for $\varepsilon_1(r) \sim 5(\log(e^e/(1-r)))^{-2}$ as $r \to 1^-$; the zeros of the solution f_1 are $z_k = 1 - \exp(e - \exp(k\pi))$, where $k \in \mathbb{Z}$. In the second case

$$A_2(z)(1-z)^2 = \frac{1}{4} + \frac{q^2}{\left(\log\frac{e}{1-z}\right)^{2(1-q)}} + \frac{1}{4}\,\frac{1-q^2}{\left(\log\frac{e}{1-z}\right)^2},$$

and so $\varepsilon_2(r) \sim 4q^2(\log(e/(1-r)))^{2(q-1)}$ as $r \to 1^-$; the zeros of the solution f_1 are $z_k = 1 - \exp\left(1 - (k\pi)^{1/q}\right)$, where $k \in \mathbb{Z}$.

4. Proof of Theorem 1

The proof is grounded on an application of a suitable family of conformal maps. The following construction, including Lemma A below, is borrowed from [6, p. 576]. Without loss of generality, we may assume $\zeta = 1$.

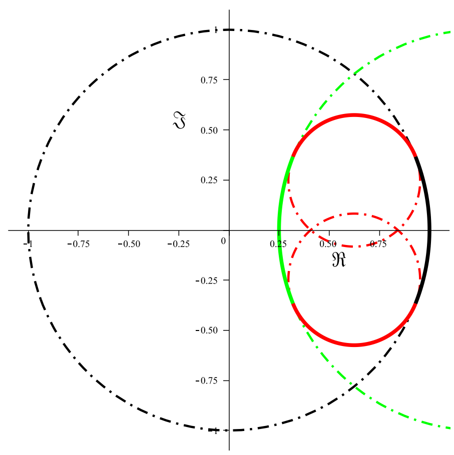

Figure 1. The boundary of $\Omega_{3/8,1/4}$ consists of the colorized bold curves.

Let $\tau, \varrho \in (0,1)$ such that $2\tau + \varrho < 1$. Consider the circles $\partial\mathbb{D}$, $\partial D(1+\varrho, 1)$ and $\partial D(c_\pm, r)$, where $c_\pm = (1+\varrho)(1 \pm i\tan\tau)/2$ and $r = |e^{i\tau} - c_+|$. The discs $D(c_\pm, r)$ are contained in both $\mathbb{D}$ and $D(1+\varrho, 1)$. Moreover, the circles $\partial D(c_\pm, r)$ intersect $\partial\mathbb{D}$ on the points $e^{\pm i\tau}$, and the common points of $\partial D(c_\pm, r)$ and $\partial D(1+\varrho, 1)$ are the reflections of $e^{\pm i\tau}$ with respect to the line $\operatorname{Re} z = (1+\varrho)/2$. Let us call them $\gamma_\pm$ according to the sign of their imaginary parts. Let $\Omega_{\tau,\varrho}$ be the Jordan domain formed by the shortest four circular arcs connecting $e^{\pm i\tau}$ and $\gamma_\pm$ on these four circles. See Figure 1 for an illustration. Let $\varphi_{\tau,\varrho}$ be the conformal map of $\mathbb{D}$ onto $\Omega_{\tau,\varrho}$. The existence of a such mapping is ensured by the Riemann mapping theorem, which also shows that under the additional conditions $\varphi_{\tau,\varrho}(0) = (1+\varrho)/2$ and $\varphi'_{\tau,\varrho}(0) > 0$ this mapping is unique.

The following lemma produces an estimate for the growth of the Schwarzian derivative of $\varphi_{\tau,\varrho}$. An alternative approach is explained in [12, pp. 198–208]: the Schwarzian derivative of $\varphi_{\tau,\varrho}$ is explicitly determined by the boundary arcs of $\Omega_{\tau,\varrho}$, since the boundary $\partial\Omega_{\tau,\varrho}$ forms a curvilinear polygon.

Lemma A ([6, Lemma 8])**.** *Let $0 < p < \infty$, and let $\tau, \varrho \in (0,1)$ such that $2\tau+\varrho < 1$. Then the function $\varphi_{\tau,\varrho}$ satisfies $(\log \varphi'_{\tau,\varrho})' \in H^p$, $\varphi''_{\tau,\varrho} \in H^p$ and*

$$\int_{\mathbb{D}} \left| \frac{\varphi''_{\tau,\varrho}(z)}{\varphi'_{\tau,\varrho}(z)} \right|^p dm(z) \longrightarrow 0, \quad \tau \to 0^+.$$

Before the proof of Theorem 1, we make some observations about $\varphi_{\tau,\varrho}$. To conclude that $\varphi_{\tau,\varrho}$ maps the open interval $(-1,1)$ into the real axis, we follow [13, p. 11]. Evidently, $\varphi_{\tau,\varrho}$ admits a Taylor expansion

$$\varphi_{\tau,\varrho}(z) = \frac{1+\varrho}{2} + a_1 z + a_2 z^2 + a_3 z^3 + \cdots,$$

where $a_1 > 0$. Define an auxiliary function $\widetilde{\varphi}(z) = \overline{\varphi_{\tau,\varrho}(\overline{z})}$. Function $\widetilde{\varphi}$ is analytic and univalent in $\mathbb{D}$, and it has a Taylor expansion

$$\widetilde{\varphi}(z) = \frac{1+\varrho}{2} + a_1 z + \overline{a}_2 z^2 + \overline{a}_3 z^3 + \cdots.$$

Note that $\widetilde{\varphi}(\mathbb{D}) = \varphi_{\tau,\varrho}(\mathbb{D})$, and hence $\widetilde{\varphi} \equiv \varphi_{\tau,\varrho}$ according to the uniqueness part of the Riemann mapping theorem. Since the Taylor expansion of $\varphi_{\tau,\varrho}$ is unique, we conclude that coefficients a_j are real for all $j \in \mathbb{N}$. This means that $\varphi_{\tau,\varrho}$ maps the interval $(-1,1)$ into the real axis, and hence is typically real. Furthermore, since $\varphi_{\tau,\varrho}$ is univalent, and as a real function of a real variable it is increasing at $z = 0$ by $\varphi'_{\tau,\varrho}(0) > 0$, we have shown that $z = 1$ is a fixed point of $\varphi_{\tau,\varrho}$. Recall that $\varphi_{\tau,\varrho}$ has an injective and continuous extension to the closed unit disc $\overline{\mathbb{D}}$ by the famous theorem of Carathéodory.

Moreover, $\varphi'_{\tau,\varrho}$ has a continuous extension to $\overline{\mathbb{D}}$, and

$$\left| \frac{\varphi''_{\tau,\varrho}(z)}{\varphi'_{\tau,\varrho}(z)} \right| (1-|z|^2)^{\frac{1}{p}} \longrightarrow 0, \quad |z| \to 1^-,$$

since both $\varphi''_{\tau,\varrho}$ and $(\log \varphi'_{\tau,\varrho})'$ belong to H^p for all $0 < p < \infty$, see [5, Theorems 3.11 and 5.9]. Standard estimates yield

$$|S_{\varphi_{\tau,\varrho}}(z)|(1-|z|^2)^{1+\frac{1}{p}} \longrightarrow 0, \quad |z| \to 1^-, \tag{4.1}$$

for all $1 \le p < \infty$.

To prove the first assertion of Theorem 1, assume that there exists $\delta > 0$ such that any non-trivial solution of (1.1) has at most one zero in $D(1,\delta)\cap\mathbb{D}$. Fix $\tau,\varrho\in(0,1)$ such that $2\tau+\varrho<1$ and $\varphi_{\tau,\varrho}(\mathbb{D})=\Omega_{\tau,\varrho}\subset D(1,\delta)\cap\mathbb{D}$. Write $T=\varphi_{\tau,\varrho}$ for short. Then, for any given linearly independent solutions f_1 and f_2 of (1.1), the meromorphic function $f_1/f_2\circ T$ is univalent in $\mathbb{D}$. Therefore

$$\left|S_{f_1/f_2}\big(T(z)\big)\big(T'(z)\big)^2+S_T(z)\right|(1-|z|^2)^2\le 6,\quad z\in\mathbb{D},\tag{4.2}$$

by Kraus theorem [9], see also [13, pp. 67–68] regarding the meromorphic case.

Let $\{w_n\}$ be any sequence of points in $\mathbb{D}$ such that $w_n\to 1$, and define z_n by the equation $T(z_n)=w_n$. Then

$$\lim_{n\to\infty} z_n=\lim_{n\to\infty}T^{-1}(w_n)=T^{-1}\Big(\lim_{n\to\infty}w_n\Big)=T^{-1}(1)=1,$$

because $z=1$ is a fixed point of T. Let L_n denote the straight line segment from $z_n\in\mathbb{D}$ to $z_n/|z_n|\in\partial\mathbb{D}$. For all n sufficiently large, we have

$$1-|T(z_n)|=|T(z_n/|z_n|)|-|T(z_n)|\le\left|\int_{L_n}T'(z)\,dz\right|\le(1-|z_n|)\sup_{z\in L_n}|T'(z)|.$$

We point out that T' is continuous in $\overline{\mathbb{D}}$, and $T'(1)\neq 0$ by the Julia-Carathéodory theorem. By the Schwarz-Pick lemma, we deduce

$$1\le\frac{1-|T(z_n)|^2}{|T'(z_n)|(1-|z_n|^2)}\le\frac{1+|T(z_n)|}{1+|z_n|}\,\frac{\sup_{z\in L_n}|T'(z)|}{|T'(z_n)|}\longrightarrow 1,\quad n\to\infty.$$

This, together with (4.1) and (4.2), yields

$$\begin{aligned}
&\limsup_{n\to\infty}|A(w_n)|(1-|w_n|^2)^2\\
&\quad=\limsup_{n\to\infty}\frac{1}{2}\left|S_{f_1/f_2}(T(z_n))\right|\big(1-|T(z_n)|^2\big)^2\\
&\quad=\limsup_{n\to\infty}\frac{1}{2}\left|S_{f_1/f_2}(T(z_n))\right||T'(z_n)|^2(1-|z_n|^2)^2\left(\frac{1-|T(z_n)|^2}{|T'(z_n)|(1-|z_n|^2)}\right)^2\le 3,
\end{aligned}$$

which is a contradiction.

5. Proof of Theorem 2

Let f be a solution of (1.1), and suppose that $0\le\delta<R<1$. We have

$$|f(z)|\le\int_\delta^{|z|}\int_\delta^t\left|f''\Big(s\tfrac{z}{|z|}\Big)\right|\,dsdt+M(\delta,f')+M(\delta,f),\quad\delta<|z|<1,$$

where $M(\delta,\,\cdot\,)$ is the maximum modulus on $|z|=\delta$. By means of (1.1), we obtain

$$\begin{aligned}
&\sup_{\delta<|z|<R}(1-|z|^2)^p|f(z)|\\
&\qquad\le\left(\sup_{\delta<|\zeta|<R}(1-|\zeta|^2)^p|f(\zeta)|\right)\left(\sup_{\delta<|\zeta|<R}(1-|\zeta|^2)^2|A(\zeta)|\right)\\
&\qquad\quad\cdot\sup_{\delta<|z|<R}\left((1-|z|^2)^p\int_\delta^{|z|}\int_\delta^t\frac{dsdt}{(1-s^2)^{p+2}}\right)+M(\delta,f')+M(\delta,f).
\end{aligned}\tag{5.1}$$

If $0 < p < \infty$, then

$$\lim_{|z|\to 1^-}\left((1-|z|^2)^p\int_\delta^{|z|}\int_\delta^t \frac{dsdt}{(1-s^2)^{p+2}}\right) = \frac{1}{4p(p+1)}$$

by the Bernoulli-l'Hôpital theorem. The estimate (5.1) implies that $\|f\|_{H_p^\infty} < \infty$ provided that $K < 4p(p+1)$. We conclude $\|f\|_{H_p^\infty} < \infty$ for any $p > (\sqrt{1+K}-1)/2$.

6. Proof of Theorem 3

To prove the first assertion of Theorem 3, suppose that f_1 is any non-trivial solution of (1.1) which has two distinct zeros $z_1, z_2 \in \mathbb{D}$ such that their hyperbolic midpoint $\xi = \xi_h(z_1,z_2)$ satisfies $1-|\xi| < 1/C$. Let $\{f_1, f_2\}$ be a solution base of (1.1). Define $h = f_1/f_2$, which implies that $S_h = 2A$.

Let $a \in \mathbb{D}$ such that $1-1/C < |a| < 1$. If we define $r_a = 1 - C^{1/2}(1-|a|)^{1/2}$, then $0 < r_a < 1$. Set $g_a(z) = (h\circ\varphi_a)(r_a z)$. Then the assumption (2.3) yields

$$\begin{aligned}|S_{g_a}(z)|(1-|z|^2)^2 &= \big|S_h\big(\varphi_a(r_az)\big)\big|\,|\varphi_a'(r_az)|^2\, r_a^2\,(1-|z|^2)^2\\ &\le 2\Big(1+C\big(1-|\varphi_a(r_az)|\big)\Big)\left(\frac{1-|z|^2}{1-r_a^2|z|^2}\right)^2 r_a^2\\ &\le 2\Big(1+C\big(1-\varphi_{|a|}(r_a)\big)\Big)\, r_a^2 \le 2, \quad z\in\mathbb{D},\end{aligned}$$

where the last inequality follows from

$$1-\Big(1+C\big(1-\varphi_{|a|}(r_a)\big)\Big)\, r_a^2 = C\,\frac{(1+r_a)(1-|a|)}{1-|a|r_a}\left(\frac{1-|a|r_a}{1-r_a} - r_a^2\right) \ge 0.$$

According to [11, Theorem 1] the function g_a is univalent in $\mathbb{D}$, and hence $h = f_1/f_2$ is univalent in the pseudo-hyperbolic disc $\Delta_p(a, r_a)$.

The argument above shows that h is univalent in $\Delta_p(\xi, r_\xi)$, and hence

$$\varrho_h(z_1,z_2) = \log\frac{1+\varrho_p(z_1,\xi)}{1-\varrho_p(z_1,\xi)} \ge \log\frac{1+r_\xi}{1-r_\xi} = \log\frac{2-C^{1/2}(1-|\xi|)^{1/2}}{C^{1/2}(1-|\xi|)^{1/2}}.$$

To prove the second assertion of Theorem 3, suppose that the hyperbolic distance between any distinct zeros $z_1, z_2 \in \mathbb{D}$ of any non-trivial solution of (1.1), for which $1-|\xi_h(z_1,z_2)| < 1/C$, satisfies (2.4) with some $0 < C < \infty$. In another words,

$$\varrho_h(z_1,z_2) \ge \log\frac{1+r_\xi}{1-r_\xi}, \quad r_\xi = 1-C^{1/2}(1-|\xi|)^{1/2}, \quad \xi = \xi_h(z_1,z_2). \tag{6.1}$$

First, we show that each non-trivial solution of (1.1) has at most one zero in

$$\Delta_h\bigg(a,\, \frac{1}{2}\log\frac{1+R_a}{1-R_a}\bigg), \quad R_a = 1-(8C)^{1/3}(1-|a|)^{1/3}, \quad 1-|a| < (8C)^{-1}.$$

Assume on the contrary that there exists a non-trivial solution having two distinct zeros $z_1, z_2 \in \Delta_p(a, R_a)$ for some $1-|a| < (8C)^{-1}$. By hyperbolic geometry we conclude $\xi \in \Delta_p(a, R_a)$, and hence

$$1-r_\xi \le C^{1/2}\left(1-\frac{|a|-R_a}{1-|a|R_a}\right)^{1/2} = \frac{C^{1/2}(1-|a|)^{1/2}\,(1+R_a)^{1/2}}{(1-|a|R_a)^{1/2}},$$

which implies

$$\frac{1+r_\xi}{1-r_\xi}\cdot\frac{1-R_a}{1+R_a}\geq\frac{(1+r_\xi)(1-|a|R_a)^{1/2}}{C^{1/2}(1-|a|)^{1/2}\,(1+R_a)^{1/2}}\cdot\frac{(8C)^{1/3}(1-|a|)^{1/3}}{1+R_a}$$
$$\geq\frac{(8C)^{1/3}}{2\sqrt{2}\,C^{1/2}}\cdot\frac{(1-R_a)^{1/2}}{(1-|a|)^{1/6}}=1.$$

We deduce

$$\varrho_h(z_1,z_2)\leq\varrho_h(z_1,a)+\varrho_h(a,z_2)<\log\frac{1+R_a}{1-R_a}\leq\log\frac{1+r_\xi}{1-r_\xi},$$

which contradicts (6.1).

Second, we derive the estimate (2.5). Let $\{f_1,f_2\}$ be a solution base of (1.1) and set $h=f_1/f_2$ so that $S_h=2A$. Set $g_a(z)=(h\circ\varphi_a)(R_az)$. Since h is univalent in each pseudo-hyperbolic disc $\Delta(a,R_a)$ for $1-|a|<(8C)^{-1}$, it follows that g_a is univalent in $\mathbb{D}$ for those values of a, and hence

$$|S_{g_a}(z)|(1-|z|^2)^2=\big|S_h\big(\varphi_a(R_az)\big)\big||\varphi_a'(R_az)|^2\,R_a^2\,(1-|z|^2)^2$$
$$=2\big|A\big(\varphi_a(R_az)\big)\big|\frac{(1-|a|^2)^2}{|1-\overline{a}R_az|^4}\,R_a^2\,(1-|z|^2)^2\leq 6,\quad z\in\mathbb{D},$$

by Kraus theorem [9]. By choosing $z=0$, we conclude

$$|A(a)|(1-|a|^2)^2\leq\frac{3}{R_a^2}=3\left(1+(1-R_a)\,\frac{1+R_a}{R_a^2}\right),\quad 1-|a|<(8C)^{-1}.$$

7. Proof of Theorem 5

We begin with the first assertion of Theorem 5. Suppose that the coefficient A satisfies (2.3) for some $0<C<\infty$, and there exists a non-trivial solution f_1 of (1.1) having two distinct zeros $z_1,z_2\in D(\zeta,(1+C)^{-1})$ for some $|\zeta|\leq C/(1+C)$. Let f_2 be a solution of (1.1) linearly independent to f_1. By setting $h=f_1/f_2$, we deduce $S_h=2A$. The Möbius transformation $T(z)=\zeta+(1+C)^{-1}z$ is a conformal map from $\mathbb{D}$ onto $D(\zeta,(1+C)^{-1})$. We proceed to prove that $g=h\circ T$ is univalent in $\mathbb{D}$. Now

$$|S_g(z)|(1-|z|^2)^2=\big|S_h\big(T(z)\big)\big|\,|T'(z)|^2(1-|z|^2)^2$$
$$\leq 2\,\frac{1+C\big(1-|\zeta+(1+C)^{-1}z|\big)}{\big(1-|\zeta+(1+C)^{-1}z|^2\big)^2}\,(1+C)^{-2}(1-|z|^2)^2,\quad z\in\mathbb{D}.$$

The proof of the first assertion is divided into two separate cases. By differentiation, there exists $0<t_C<1/3$ such that the auxiliary function

$$\mu(t)=\big(1+C(1-t)\big)(1-t^2)^{-2},\quad 0<t<1,$$

is decreasing for $0<t<t_C$, and increasing for $t_C<t<1$.

(i) Suppose that $z\in\mathbb{D}$, and $|\zeta+(1+C)^{-1}z|>t_C$. By the triangle inequality $|\zeta+(1+C)^{-1}z|\leq(1+C)^{-1}(C+|z|)$, and hence

$$|S_g(z)|(1-|z|^2)^2\leq 2\,\frac{1+C\big(1-(1+C)^{-1}(C+|z|)\big)}{\big(1-(1+C)^{-2}(C+|z|)^2\big)^2}\,(1+C)^{-2}(1-|z|^2)^2$$
$$=2\,\frac{(1+2C-C|z|)(1+C)(1+|z|)^2}{(|z|+2C+1)^2}\leq 2. \tag{7.1}$$

The inequality in (7.1) follows by differentiation, since the quotient is an increasing function of $|z|$ for $0 < |z| < 1$.

(ii) Suppose that $z \in \mathbb{D}$, and $|\zeta + (1-a)z| \leq t_C$. Since $\mu(0) \geq \mu(t)$ for all $0 < t \leq t_C$, we deduce

$$|S_g(z)|(1-|z|^2)^2 \leq \frac{2}{1+C} < 2.$$

By means of (i), (ii) and [11, Theorem 1], we conclude that g is univalent in $\mathbb{D}$. This is a contradiction, since the preimages $T^{-1}(z_1) \in \mathbb{D}$ and $T^{-1}(z_2) \in \mathbb{D}$ are distinct zeros of g. The first assertion of Theorem 5 follows.

We turn to consider the second assertion of Theorem 5. Suppose that there exists $0 < C < \infty$ such that any non-trivial solution of (1.1) has at most one zero in any Euclidean disc $D(\zeta, (1+C)^{-1})$ for $|\zeta| \leq C/(1+C)$. Let $\{f_1, f_2\}$ be a solution base of (1.1), and set $h = f_1/f_2$ which implies $S_h = 2A$. Suppose that $a \in \mathbb{D}$ and $|a| > C/(1+C)$. Set $g_a(z) = (h \circ \varphi_a)(r_a z)$, where

$$r_a^2 = \frac{|a| - \frac{C}{1+C}}{|a|\left(1 - |a|\frac{C}{1+C}\right)}, \quad 0 < r_a < 1.$$

Then

$$\Delta_p(a, r_a) = D\left(\frac{a}{|a|} \cdot \frac{C}{1+C}, \frac{r_a(1-|a|^2)}{1-r_a^2|a|^2}\right) \subset D\left(\frac{a}{|a|} \cdot \frac{C}{1+C}, \frac{1}{1+C}\right).$$

Since $z \mapsto \varphi_a(r_a z)$ maps $\mathbb{D}$ onto $\Delta_p(a, r_a)$, it follows that g_a is univalent in $\mathbb{D}$ by the assumption. Hence

$$\begin{aligned}
|S_{g_a}(z)|(1-|z|^2)^2 &= \left|S_h\big(\varphi_a(r_a z)\big)\right| |\varphi_a'(r_a z)|^2 \, r_a^2 \, (1-|z|^2)^2 \\
&= 2\left|A\big(\varphi_a(r_a z)\big)\right| \left(\frac{1-|a|^2}{|1-\overline{a} r_a z|^2}\right)^2 r_a^2 \, (1-|z|^2) \leq 6, \quad z \in \mathbb{D},
\end{aligned}$$

by Kraus theorem [9]. By choosing $z = 0$, we obtain

$$|A(a)|(1-|a|^2)^2 \leq \frac{3}{r_a^2} = 3\left(1 + \frac{\frac{C}{1+C}(1+|a|)}{|a| - \frac{C}{1+C}}(1-|a|)\right), \quad \frac{C}{1+C} < |a| < 1.$$

8. Proof of Theorem 6

If A is analytic in $\mathbb{D}$, and satisfies (2.3) for some $0 < C < \infty$, then any non-trivial solution f of (1.1) has at most one zero z_n in any horodisc

$$\mathcal{D}_\theta = D\left(e^{i\theta}\frac{C}{1+C}, \frac{1}{1+C}\right), \quad e^{i\theta} \in \partial\mathbb{D},$$

by Theorem 5. Suppose that $0 < 1 - r < 2/(1+C)$. Now $r \in \partial\mathcal{D}_\theta$ if and only if

$$\left|e^{i\theta}\frac{C}{1+C} - r\right|^2 = \frac{1}{(1+C)^2}.$$

The positive solution $\theta = \theta(r)$ of this equation satisfies

$$\theta(r) = \arccos\frac{C - 1 + r^2(1+C)}{2rC} \sim \sqrt{2/C}\,(1-r)^{1/2}, \quad r \to 1^-.$$

This implies that the zeros z_n of f, for which $0 < 1 - |z_n| < 2/(1+C)$, induce pairwise disjoint zero-free tent-like domains $\Omega_n \subset \mathbb{D}$, which intersect $\partial\mathbb{D}$ on arcs $I_n = \overline{\Omega}_n \cap \partial\mathbb{D}$ of normalized length

$$\ell(I_n) \sim \frac{\sqrt{2}}{\pi\sqrt{C}}\,(1-|z_n|)^{1/2}, \quad n \to \infty.$$

Consequently, if Q is any Carleson square for which $\ell(Q) < 2/(1+C)$, then

$$\sum_{z_n \in Q} (1-|z_n|)^{1/2} \lesssim \sum_{z_n \in Q} \ell(I_n) \lesssim \ell(Q)^{1/2},$$

where the comparison constants depend on $0 < C < \infty$. By a standard argument, this implies (2.6) for any Carleson square Q.

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